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MASS TRANSFER IN A PULSATING BUBBLE*

V.S. BERMAN and A.D. POLYANIN

Mass transfer between a pulsating bubble and a surrounding medium at large and small Peclet numbers is considered. The dependence of the Sherwood number on time is found for an arbitrary periodic law of variation of the bubble radius. The case of sinusoidal oscillations is studied in detail.

1. Dynamics of a pulsating bubble. The spherically symmetric oscillations of a bubble under various conditions have been studied in many publications (e.g. /1-7/). Let us list here the fundamental properties of such motions, which will be of use later when analysing mass transfer in a pulsating bubble.

The radial component of the velocity of the fluid outside the bubble is described by the expression

$$v_r = R^2R'/r^2, \quad R' = dR/dt_* \tag{1.1}$$

Here r is the radial coordinate, t_* is the time and $R = R(t_*)$ is the law of motion of the bubble boundary, which can be found, under fairly general assumptions, by solving the differential equation /1-5/

$$\rho(RR'' + {}^3/_2R'^2) + 4\mu R'/R = g_*(R) + \varphi_*(t_*) \tag{1.2}$$

where μ and ρ is the dynamic viscosity and the density of the surrounding medium.

In order to complete the formulation of the problem we must supplement Eq.(1.2) by the initial conditions $R(0) = R_0, R'(0) = 0$ where R_0 is the initial radius of the bubble. (Sometimes a periodic solution of Eq.(1.2) has to be found).

The function g_* in (1.2) is usually chosen in the form /1-5/

$$g_*(R) = p_{g0}(R_0/R)^{3\gamma} - p_\infty - 2\sigma/R \tag{1.3}$$

where p_∞ is the static pressure at infinity, σ is the surface tension, γ is the ratio of the specific heats and p_{g0} is a constant whose dimensions are that of pressure.

In the case of thin elastic spherical shells (e.g. a rubber ball) oscillating in a liquid or gas, a linear function R /8/ must be subtracted from the right-hand side of the expression (1.3) when $\sigma = 0$.

In the case of forced oscillation of the bubble, φ_* in (1.2) is a T_*-periodic function and is responsible for the variation in the pressure field. In this case we can assumed without loss of generality that the following condition holds:

$$\langle\varphi_*\rangle \equiv \frac{1}{T_*}\int_0^{T_*}\varphi_*(t_*)\,dt_* = 0$$

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In dimensionless variables Eq. (1.2) becomes

$$aa^{\cdot\cdot} + {}^3/_2 a^{\cdot 2} + \beta a^{\cdot}/a = g(a) + \varphi(t) \tag{1.4}$$

$$a = \frac{R}{R_0}, \quad t = \frac{t_*}{R_0}\sqrt{\frac{p_s}{\rho}}, \quad a^{\cdot} = \frac{da}{dt}, \quad \beta = \frac{4\mu}{R_0\sqrt{\rho p_s}}$$

$$g(a) = g_*(R)/p_s, \quad \varphi(t) = \varphi_*(t_*)/p_s$$

where p_s is a constant chosen as the pressure scale.

Free oscillations of a bubble in an ideal fluid. When $\beta = 0$ and $\varphi = 0$, Eq. (1.4) can be integrated in quadrature. A single integration of (1.4) yields /3/

$$a^{\cdot 2} = \frac{2}{a^3}G(a), \quad G(a) = \int_1^a g(x)x^2\,dx \tag{1.5}$$

where $a \geqslant 1$ when $g(1) \geqslant 0$ and $a < 1$ when $g(1) < 0$.

Integrating (1.5) we obtain an implicit form of the relation connecting the variation in the region of the bubble with time

$$t = \theta(a), \quad \theta(a) = \frac{1}{\sqrt{2}}\left|\int_0^a \frac{x^{3/2}\,dx}{\sqrt{G(x)}}\right| \tag{1.6}$$

The initial conditions $a(0) = 1, a^{\cdot}(0) = 0$ were taken into account in deriving formulas (1.5) and (1.6).

We further assume that the function g is monotonic and has a unique root a_e: $g(a_e) = 0$ where $a_e > 1$ when $g(1) > 0$ and $0 < a_e < 1$ when $g(1) < 0$. In this case the bubble will oscillate between the extremal values $a = 1$ and $a = \bar{a}$ where $\bar{a} \neq 1$ is the root of the equation $G(\bar{a}) = 0$. The period of oscillations of the bubble is found from the formula $T = 2\theta(\bar{a})$, which takes into account both possible situations: $\bar{a} < a_e < 1$ and $1 < a_e < \bar{a}$.

Forced oscillation of a bubble in a viscous fluid near the position of equilibrium. We shall now consider the pulsation of a bubble in a field of variable pressure, with $\max|\varphi| \ll 1$. It will be convenient to use the equilibrium radius of the bubble as the scale of the length, with the radius determined by solving the algebraic equation $g_*(R_0) = 0$. We choose, as the scale of the pressure, the quantity $p_s = R_0|\partial g_*/\partial R|_{R=R_0}$ (we assume that g_* is a monotonically decreasing function of R).

When the dimensionless variables are defined in this manner, Eq. (1.4) will have, when $\varphi = 0$ and by virtue of the property $g(1) = 0$, a stationary solution $a = 1$. Linearizing (1.4) near this point we obtain the following equation for the forced oscillations of the bubble:

$$y^{\cdot\cdot} + \beta y^{\cdot} + y = \varphi(t), \quad y = a - 1 \tag{1.7}$$

When $\beta > 0$ any solution of Eq. (1.7) will, as $t \to \infty$, approach the periodic mode with the same period as that of the function φ. In the case of sinusoidal oscillations, the corresponding solution will have the form

$$y = \frac{A}{(1-\omega^2)^2 + \omega^2\beta^2}\left[(1-\omega^2)\sin\omega t - \omega\beta\cos\omega t\right] \tag{1.8}$$

In this special case y differs from the function φ only in the phase shift.

Forced oscillations of a bubble in the case of low- and high-frequency variations in pressure. When the external pressure undergoes low-frequency changes, the function $\varphi(t)$ in (1.4) will have a long period $T \gg 1$. In this case we can neglect the derivatives in (1.4), and we obtain a relation for the time-dependence of the radius of the bubble by solving the algebraic equation $g(a) = -\varphi(t)$.

The high-frequency pressure oscillations correspond to small values of the period of the function φ. When $T \ll 1$, we find it convenient to use the new variables $\tau = t/T$, $x = a^{5/2}$, which enables us to write Eq. (1.4) in the following form:

$$\frac{d^2x}{d\tau^2} + T\beta x^{-4/5}\frac{dx}{d\tau} = \frac{5}{2}T^2 x^{1/5}\left[g(x^{2/5}) + \psi(\tau)\right] \tag{1.9}$$

Here $\psi(\tau) \equiv \varphi(t)$ is a function with a unit period: $\psi(\tau) = \psi(\tau + 1)$.

We construct the periodic solution of Eq. (1.9) using a regular expansion in the small parameter T. Taking into account the property $g(1) = 0$, we can obtain the relation $x \approx 1 + T^2 x_2$ where x_2 is the periodic solution of the equation $dx^2_2/d\tau^2 = {}^5/_2\psi(\tau)$. From this we obtain the law of motion of the bubble boundary $a = x^{2/5} \approx 1 - A\omega^{-2}\sin\omega t$ for sinusoidal oscillations $\varphi = A\sin\omega t$ as $\omega = 2\pi/T \to \infty$

Equations of the dynamics of a spherical bubble in a viscous fluid, integrable in quadratures. Let us now describe some equations of the form (1.4) integrable in quadratures when $\varphi = 0$ and $\beta \neq 0$.

Using the standard substitution $a^{\cdot} = u$ $(a^{\cdot\cdot} = u du/da)$, we reduce the order of Eq.(1.4). Passing now to the new variables $z = \sqrt{a}$, $w = a^{3/2}u$ $(u = a^{\cdot})$, we obtain the Abel equation

$$w dw/dz = -2\beta w + 2z^5 g(z^2) \tag{1.10}$$

By virtue of the initial condition $a = 1$, $a^{\cdot} = 0$ at $t = 0$, the quantity sought must satisfy the condition that $w = 0$ when $z = 1$.

In particular, when the function g_* is given by formula (1.3), Eq.(1.10) takes the following form when $\sigma = 0$, $p_\infty = 0$:

$$w dw/dz = -2\beta w + 2z^{5-6\gamma} \tag{1.11}$$

Here $p_s = p_{g0}$ is chosen as the characteristic quantity.

Eq.(1.11) is an equation with separable variables when $\gamma = {}^5/_6$, and becomes homogeneous when $\gamma = {}^2/_3$. In the isothermal case when $\gamma = 1$, we can obtain from (1.11) by making the substitution $\zeta = 1/z$, $v = w + 2\beta z$, the linear equation $2d\zeta/dv = -v\zeta + 2\beta$. In all these cases Eq.(1.11) can be easily integrated. Moreover, the solution of Eq.(1.11) when $\gamma = {}^{11}/_{12}$ and $\gamma = {}^7/_6$ can be written in terms of Bessel functions of order ${}^1/_3$.*(*Polyanin A.D. Abel equations and related equations of non-linear mechanics integrable in quadratures. Preprint 271. Moscow, Inst. Problem Mekhaniki, Akad. Nauk SSSR, 1986.)

2. Formulation of the problem of mass transfer in a pulsating bubble.

Let us now consider the diffusion of material dissolved in a fluid to the surface of a pulsating bubble.

We know /1, 2/ that the change in the volume of the bubble due to diffusion processes takes place very slowly. Therefore, we shall assume that the time-dependence of the radius of the bubble is given by the function $R = R(t_*)$ satisfying the condition $R_{\min} < R < R_{\max}$, where $R_{\min}/R_{\max} = O(1)$. The oscillations of the bubble can be caused, for example, by periodic variation of the external pressure (specific examples of this type were discussed in Sect.1).

We assume that at the surface of the bubble and away from it the concentration takes constant values of zero and C_∞, respectively, and we can disregard the presence of diffusing matter within the bubble.

The distribution of the concentration within the fluid is described by the equation of convective diffusion and boundary conditions

$$\frac{\partial C}{\partial t_*} + v_r \frac{\partial C}{\partial r} = D \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial C}{\partial r}\right) \tag{2.1}$$

$$r = R(t_*),\quad C = 0;\quad r \to \infty,\quad C \to C_\infty \tag{2.2}$$

Here C is the concentration within the continuous phase, D is the coefficient of diffusion and v_r is the radial component of the velocity of the fluid, determined from formula (1.1).

Referring the radial coordinate r in (2.1) and (2.2) to the bubble radius R we arrive at the problem with fixed boundaries, which has the following form in dimensionless coordinates:

$$a^2 \frac{\partial c}{\partial t} + a \frac{da}{dt}\left(\frac{1}{\xi^2} - \xi\right)\frac{\partial c}{\partial \xi} = \frac{1}{P}\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial c}{\partial \xi}\right) \tag{2.3}$$

$$\xi = 1,\quad c = 1;\quad \xi \to \infty,\quad c \to 0 \tag{2.4}$$

$$\xi = \frac{r}{R(t_*)},\quad t = \frac{t_*}{T_*},\quad c = \frac{C_\infty - C}{C_\infty},\quad P = \frac{R_0^2}{T_* D}$$

Here R_0 is the initial radius of the bubble, P is the Peclet number, T_* is the characteristic time of oscillation (in the case of free oscillations of the bubble the quantity $R_0\sqrt{\rho/p_s}$ can be taken as T_*, see Eq.(1.4)).

Eq.(2.3) and boundary conditions (2.4) must be supplemented by the initial condition. Two situations are of interest.

Let there be no pulsations when $t \leqslant 0$. The corresponding stationary solution of problem (2.3), (2.4) when $a = 1$ is given by the expression $c = 1/\xi$. In this case the initial condition will have the form

$$t = 0,\quad c = 1/\xi \tag{2.5}$$

When the bubble oscillates periodically, it makes sense to seek a periodic solution of problem (2.3), (2.4).

The most important characteristic of the mass transfer between the bubble and the surrounding fluid is the Sherwood number, which can be calculated using the formula

$$\mathrm{Sh} = I/(4\pi R D C_\infty) = -(\partial c/\partial \xi)_{\xi=1} \tag{2.6}$$

where I is the dimensional magnitude of the total diffusion flux.

We shall construct the asymptotic solutions of the problem of mass transfer between the pulsating bubble and the surrounding medium for large and small Peclet numbers.

3. Large Peclet numbers ($P \gg 1$). We pass, in Eq.(2.3) from the variables t, ξ to the new variables t, η, where the relation $\eta = \eta(t, \xi)$ will be determined later. As a result we have

$$a^2 \frac{\partial c}{\partial t} + \left\{a^2 \frac{\partial \eta}{\partial t} + a \frac{da}{dt}\left(\frac{1}{\xi^2} - \xi\right)\frac{\partial \eta}{\partial \xi}\right\}\frac{\partial c}{\partial \eta} = \frac{1}{P}\frac{1}{\xi^2}\frac{\partial \eta}{\partial \xi}\frac{\partial}{\partial \eta}\left(\xi^2 \frac{\partial \eta}{\partial \xi}\frac{\partial c}{\partial \eta}\right) \tag{3.1}$$

We shall require that the function η will cause the expression within the braces to vanish. This condition leads to a first-order linear partial differential equation in η, whose solution is given by the formula

$$\eta = a(\xi^3 - 1)^{1/3}, \quad a = a(t) \tag{3.2}$$

Substituting relation (3.2) into (3.1) and (2.4), we obtain the following equation and boundary conditions

$$\frac{\partial c}{\partial t} = \frac{1}{P}\frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left[\frac{(\eta^3 + a^3)^{4/3}}{\eta^2}\frac{\partial c}{\partial \eta}\right] \tag{3.3}$$

$$\eta = 0, \quad c = 1; \quad \eta \to \infty, \quad c \to 0 \tag{3.4}$$

We shall construct the asymptotic solution as $P \to \infty$, describing the process in the whole interval $0 \leqslant t < \infty$, on the basis of a two-scale temporal expansion /9, 10/. To do this, we shall introduce the additional variable

$$\tau = t/P \tag{3.5}$$

and we will seek the distribution of concentration over the whole region $0 \leqslant \eta < \infty$ in the form of an expansion in inverse powers of the Peclet number

$$c = c_0(\eta, t, \tau) + P^{-1}c_1(\eta, t, \tau) + \ldots, \quad c_1/c_0 = O(1) \tag{3.6}$$

Introducing two different time scales increases the number of independent variables. Therefore, the time derivative should be calculated using the rule for differentiating a composite function using the formula $\partial/\partial t = \partial/\partial t + P^{-1}\partial/\partial\tau$.

Taking into account what was said above, we substitute expansion (3.6) into (3.3). Equating coefficients of like powers of P, we obtain the equations

$$\partial c_0/\partial t = 0 \quad (\eta = 0, \ c_0 = 1; \ \eta \to \infty, \ c_0 \to 0) \tag{3.7}$$

$$\frac{\partial c_1}{\partial t} = \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left[\frac{(\eta^3 + a^3)^{4/3}}{\eta^2}\frac{\partial c_0}{\partial \eta}\right] - \frac{\partial c_0}{\partial \tau} \tag{3.8}$$

Eqs.(3.7) with the corresponding boundary and initial conditions are found to be insufficient to determine the principal term of the expansion for the concentration. We can obtain from (3.7) only the general form of the solution needed

$$c_0 = c_0(\eta, \tau) \tag{3.9}$$

To obtain the necessary additional information on the function c_0, we shall use the equation for the next term of the expansion, c_1 (3.8). Taking (3.9) into account we obtain the general solution of Eq.(3.8)

$$c_1(\eta, t, \tau) = t\left\{\frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left(\frac{1}{\eta^2}\left[\frac{1}{t}\int_0^t (\eta^3 + a^3)^{4/3}\,dt\right]\frac{\partial c_0}{\partial \eta}\right) - \frac{\partial c_0}{\partial \tau}\right\} + B(\eta, \tau) \tag{3.10}$$

In order to make the expansion (3.6) uniformly applicable over the whole interval $0 \leqslant t < \infty$, the ratio c_1/c_0 must be bounded as $t \to \infty$. Expressions (3.9) and (3.10) imply that this condition will hold only when the zeroth term of the expansion c_0 satisfies the equation

$$\frac{\partial c_0}{\partial \tau} = \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left[\frac{f(\eta)}{\eta^2}\frac{\partial c_0}{\partial \eta}\right] \tag{3.11}$$

$$\eta = 0, \quad c_0 = 1; \quad \eta \to \infty, \quad c_0 \to 0$$

where the positive function f is given by

$$f(\eta) = \lim_{t \to \infty}\frac{1}{t}\int_0^t (\eta^3 + a^3)^{4/3}\,dt, \quad a = a(t) \tag{3.12}$$

Eq.(3.11) has a stationary solution

$$c_0^{st} = \frac{J(\eta)}{J(0)}, \quad J(\eta) = \int_\eta^\infty \frac{\eta^2}{f(\eta)}\, d\eta \tag{3.13}$$

The corresponding Sherwood number, which is calculated using the formula (2.6), taking expressions (3.2) and (3.13) into account, is equal to

$$\mathrm{Sh} = a^3(t)/[f(0)J(0)] \tag{3.14}$$

It is important to note that the solution of Eq.(3.11), as $\tau \to \infty$, will approach the "stationary" mode (3.13) irrespective of the initial condition. In the case of periodic oscillations of the bubble, relations (3.13) and (3.2) will correspond to the periodic solution of the problem.

In case (2.5), the initial condition for the principal term of the expansion of the concentration will have the form

$$\tau = 0, \quad c_0 = (\eta^3 + 1)^{-1/3} \tag{3.15}$$

We now consider some examples.

1°. Let $a(t) \to 1$ as $t \to \infty$. This condition holds, for example, when the bubble oscillates for a limited period. (This situation occurs when a shock wave passes through the fluid). In this case formula (3.12) will yield

$$f(\eta) = (\eta^3 + 1)^{4/3} \tag{3.16}$$

Substituting (3.16) into expression (3.13), we obtain $c_0 = (\eta^3 + 1)^{-1/3}$. We see that this solution satisfies the initial condition (3.15).

We calculate the Sherwood number using formula (3.14), taking into account (3.16). This yields the following simple relation:

$$\mathrm{Sh} = a^3(t) \tag{3.17}$$

2°. When the oscillations of the bubble are periodic, relation (3.12) simplifies and takes the form

$$f(\eta) = \frac{1}{T}\int_0^T (\eta^3 + a^3)^{4/3}\, dt \tag{3.18}$$

where T is the period of the function a.

Let us consider the small-amplitude oscillations of the bubble

$$a(t) = 1 + \delta(t); \quad \max|\delta| \ll 1, \quad \langle\delta\rangle \equiv \frac{1}{T}\int_0^T \delta\, dt = 0 \tag{3.19}$$

Substituting (3.19) into (3.14), we obtain the functions and integrals appearing in formula (3.14):

$$f(\eta) = (1 + \eta^3)^{4/3}\left\{1 + \frac{2(3 + 2\eta^3)}{(1 + \eta^3)^2}\langle\delta^2\rangle\right\}, \quad f(0) = 1 + 6\langle\delta^2\rangle$$
$$J(0) = 1 - \frac{9}{7}\langle\delta^2\rangle$$

As a result, we have the following expressions for the Sherwood number and mean Sherwood number over a single period of oscillation of the bubble:

$$\mathrm{Sh} = 1 + 3\delta + 3\delta^2 - \frac{33}{7}\langle\delta^2\rangle + o(\langle\delta^2\rangle) \tag{3.20}$$

$$\langle\mathrm{Sh}\rangle = 1 - \frac{12}{7}\langle\delta^2\rangle + o(\langle\delta^2\rangle) \tag{3.21}$$

In the case of a sinusoidal law of variation of the radius with time, which occurs during forced oscillations of the bubble (see Sect.1) and corresponds to $\delta = \varepsilon \sin t$, $\varepsilon \ll 1$ in (3.19), we obtain for the mean Sherwood number (3.21) the expression $\langle\mathrm{Sh}\rangle = 1 - {}^6/_7\varepsilon^2 + o(\varepsilon^2)$.

Here we must mention an interesting circumstance. From formula (3.21) it follows that when periodic oscillations of the bubble are small, the mean Sherwood number is less than unity. At the same time it can be shown that the mean (over the period of the oscillations) total diffusion flux towards the surface of the oscillating bubble will be greater than that in the case of a bubble at rest. (This assertion can be proved by multiplying the right-hand side of formula (3.20) by a^2 and averaging over one period of the oscillations).

4. Small Peclet numbers ($P \ll 1$). We shall now construct an approximate analytic solution of problem (2.3)-(2.5) using the method of matching the asymptotic expansions /9-11/ in small Peclet numbers.

When $\xi = O(P^{-1/2})$, the terms on the right-hand side of (2.3) will be of the same order as those on the left-hand side. Therefore, we must separate out two regions: the inner region $\Omega^{(i)} = \{1 \leqslant \xi \leqslant O(P^{-1/2})\}$ and the outer region $\Omega^{(e)} = \{O(P^{-1/2}) \leqslant \xi\}$. In the inner region we retain the previous variables ξ, t, and in the outer region we replace ξ by a compressed radial coordinate

$$y = P^{1/2}\xi \tag{4.1}$$

We will seek the inner and outer expansions of the concentration, respectively, in the form

$$c = \frac{1}{\xi} + \sum_{n=1}^{\infty} \varepsilon_n^{(i)}(P)\, c_n^{(i)}(\xi, t) \quad \text{in} \quad \Omega^{(i)} \tag{4.2}$$

$$c = \sum_{n=1}^{\infty} \varepsilon_n^{(e)}(P)\, c_n^{(e)}(y, t) \quad \text{in} \quad \Omega^{(e)} \tag{4.3}$$

$$(\varepsilon_{n+1}^{(i)}/\varepsilon_n^{(i)} \to 0, \quad \varepsilon_{n+1}^{(e)}/\varepsilon_n^{(e)} \to 0 \quad \text{as} \quad P \to 0)$$

The terms of the inner expansion are found from Eq.(2.3), and they satisfy the homogeneous boundary and initial conditions (this follows from (2.4) and (2.5)):

$$\xi = 1, \quad c_n^{(i)} = 0; \quad t = 0, \quad c_n^{(i)} = 0 \quad (n = 1, 2, \ldots) \tag{4.4}$$

The outer expansion is obtained from the equation

$$a^2 \frac{\partial c}{\partial t} + a \frac{da}{dt} \left(\frac{P^{3/2} - y^3}{y^2}\right) \frac{\partial c}{\partial y} = \frac{1}{y^2} \frac{\partial}{\partial y} \left(y^2 \frac{\partial c}{\partial y}\right) \tag{4.5}$$

$$y \to \infty, \quad c_n^{(i)} \to 0 \quad (n = 1, 2, \ldots)$$

which is obtained from (2.3) by making the substitution (4.1).

The conditions for the matching of the principal terms of the outer expansion (4.3) and inner expansion (4.2) yield, taking into account (2.5) and (4.1), an explicit expression for the coefficient $\varepsilon_1^{(e)} = P^{1/2}$, as well as the initial and boundary conditions for $c_1^{(e)}$:

$$t = 0, \quad c_1^{(e)} = 1/y; \quad y \to 0, \quad c_1^{(e)} \to 1/y; \quad y \to \infty, \quad c_1^{(e)} \to 0 \tag{4.6}$$

Substituting (4.3) into (4.5) we obtain the equation for $c_1^{(e)}$:

$$a^2 \frac{\partial c_1^{(e)}}{\partial t} - ya \frac{da}{dt} \frac{\partial c_1^{(e)}}{\partial y} = \frac{1}{y^2} \frac{\partial}{\partial y} \left(y^2 \frac{\partial c_1^{(e)}}{\partial y}\right) \tag{4.7}$$

Changing (4.6) and (4.7) from the variables $y, t, c_1^{(e)}$ to new variables z, t, u, where

$$z = a(t)y, \quad u = zc_1^{(e)} \tag{4.8}$$

we obtain the problem for the usual parabolic equation

$$\partial u/\partial t = \partial^2 u/\partial\ {}^2; \quad t = 0, \quad u = 1; \quad z = 0, \quad u = a(t)$$

whose solution is well-known. Therefore the principal term of the outer expansion is given by the expression

$$c_1^{(e)} = \frac{1}{z} \operatorname{erf}\left(\frac{z}{2\sqrt{t}}\right) + \frac{1}{2\sqrt{\pi}} \int_0^t \frac{a(t')}{(t-t')^{3/2}} \exp\left[-\frac{z^2}{4(t-t')}\right] dt' \tag{4.9}$$

When $y \to 0$, formula (4.9), taking (4.8) into account, yields

$$y \to 0, \quad c_1^{(e)} = \frac{1}{y} - \frac{1}{\sqrt{\pi}} \int_0^t \frac{da}{dt'} \frac{dt'}{\sqrt{t-t'}} + \frac{1}{2} ya \frac{da}{dt} + o(y) \tag{4.10}$$

Let us substitute (4.10) into (4.3) and change to the inner variable $\xi = P^{-1/2}y$. The matching condition yields the coefficients of the series (4.2):

$$\varepsilon_1^{(i)} = P^{1/2}, \quad \varepsilon_2^{(i)} = P \tag{4.11}$$

Let us substitute the expansion (4.2), (4.11) into (2.3) and separate terms of like powers of P. This yields the equations

$$\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial c_1^{(i)}}{\partial\xi}\right)=0;\quad \frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial c_2^{(i)}}{\partial\xi}\right)=a\frac{da}{dt}\frac{\xi^3-1}{\xi^2}$$

whose general solutions, satisfying the boundary conditions (4.4), are given by the expressions

$$c_1^{(i)}=E(t)\left(\frac{1}{\xi}-1\right) \tag{4.12}$$

$$c_2^{(i)}=F(t)\left(\frac{1}{\xi}-1\right)+\frac{1}{2}a\frac{da}{dt}\left(\xi-\frac{1}{\xi^2}\right) \tag{4.13}$$

Matching the series (4.2) and (4.3) and taking into account (4.10) and (4.12), we obtain

$$E(t)=\frac{1}{\sqrt{\pi}}\int_0^t\frac{da}{dt'}\frac{dt'}{\sqrt{t-t'}} \tag{4.14}$$

Using formula (4.12), we can establish that $\varepsilon_2^{(e)}=P$. This enables us to use standard methods to formulate the problem for the function $c_2^{(e)}$, which will differ from problem (4.7), (4.6) only in the initial condition ($t=0,\ c_2^{(e)}=0$) and boundary condition ($y\to 0,\ c_2^{(e)}\to E(t)/y$). The solution of the problem for the function $c_2^{(e)}$ has the form

$$c_2^{(e)}=\frac{1}{2\sqrt{\pi}}\int_0^t\frac{a(t')E(t')}{(t-t')^{3/2}}\exp\left[-\frac{a^2(t)y^2}{4(t-t')}\right]dt'$$

Let us expand this expression as $y\to 0$, and change to the inner variable. As a result of the matching we obtain the boundary condition as $\xi\to\infty$ for $c_2^{(i)}$, and this enables us to obtain

$$F(t)=\frac{1}{\sqrt{\pi}}\int_0^t\frac{d}{dt'}[a(t')E(t')]\frac{dt'}{\sqrt{t-t'}} \tag{4.15}$$

Substituting expressions (4.12)-(4.15) into the formula (2.6), we obtain the relation connecting the Sherwood number with time

$$\mathrm{Sh}=1+P^{1/2}E(t)+P[F(t)-{}^3/_2a\,da/dt]+o(P) \tag{4.16}$$

We shall consider, as an example, a sinusoidal law of oscillation of the bubble

$$a(t)=1+\alpha\sin t,\quad \alpha=O(1) \tag{4.17}$$

Substituting (4.17) into the integrand in (4.14), we obtain

$$E(t)=\alpha\sqrt{2}\,[C(\sqrt{t})\cos t+S(\sqrt{t})\sin t] \tag{4.18}$$

$$C(x)=\sqrt{\frac{2}{\pi}}\int_0^x\cos\zeta^2\,d\zeta,\quad S(x)=\sqrt{\frac{2}{\pi}}\int_0^x\sin\zeta^2\,d\zeta$$

$$(C(\infty)=S(\infty)={}^1/_2)$$

Formulas (4.16) and (4.18) yield the periodic dependence of the Sherwood number on time as $t\to\infty$:

$$\mathrm{Sh}=1+P^{1/2}\sin(t+{}^1/_4\pi)+O(P)$$

i.e. the diffusion flux against the bubble, in this approximation, suffers a phase shift of $\pi/4$.

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DETERMINATION OF THE EQUILIBRIUM SHAPE OF THE BODIES FORMED DURING THE SOLIDIFICATION OF FILTRATION FLOW*

K.G. KORNEV and V.A. CHUGUNOV

It is shown that the problem of determining the equilibrium shape of the bodies formed when a filtration flow solidifies, can be reduced to the Riemann problem with shear. A solitary body is used as an example, and an algorithms for determining its boundary is constructed and realized. The qualitative properties of the solution of the problem in question are studied.

1. Formulation of the problem. The method of freezing water-laden rocks is widely used in building various types of constructions /1/. The process of solidifying a filtration flow around a cold source is characterized by the fact that after a time a thermal balance is reached. The heat flux densities at the phase boundary become equal, and this means that the rate of formation of the solid becomes equal to zero. Thus the shape of the solid formed when the filtration flow solidifies reaches, in time, its limiting form, which we shall call the equilibrium form.

If we assume that the process takes place in the plane $z = x + iy$, that the filtration obeys D'Arcy's law, that the fluid is incompressible and that the thermophysical characteristics of the filtering medium are constant, the mathematical model of the phenomenon in question can be represented in the form

$$\mathbf{v} = -k\nabla p, \quad \operatorname{div}\mathbf{v} = 0, \quad K_c\mathbf{v}\nabla t = a_+\Delta t \quad z \in D \tag{1.1}$$
$$\Delta t_k = 0, \quad z \in D_k$$

$$\mathbf{v} \to v_\infty, \quad t \to t_\infty, \quad |z| \to \infty \tag{1.2}$$

$$\lambda_+ \partial t/\partial n = \lambda_- \partial t_k/\partial n, \quad t = t_k = t_*, \quad z \in \partial D_k \tag{1.3}$$

$$t_k = t_0 < t_*, \quad z \in \Gamma_k \tag{1.4}$$

Here D is the region of filtration, D_k is the region occupied by the solid, ∂D_k is its boundary, $\mathbf{v}$ is the rate of filtration, p is pressure, k is the coefficient of filtration, t, t_k are the temperatures in the region D and D_k respectively, K_c is the ratio of the heat capacities of the liquid and the filtering medium, λ_+, λ_- are the thermal conductivities in the regions D and D_k respectively, a_+ is the thermal diffusivity in D, n is the normal to the surface ∂D_k, external with respect to the region D_k, t_* is the temperature at

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